University of Saskatchewan
Department of Mathematics and Statistics
Math 224 (02, G.Patrick)

Friday February 24, 2006 | Test #1 | 60 minutes

This examination consists of two parts. Part A contains short, routine questions, which you should answer fully but succinctly in the space provided. The questions in Part B are more difficult, and some are designed to challenge you. Fully answer all questions of Part B in the space provided.

You should complete Part A rapidly, and save about half your time to answer the questions in Part B. Part A is worth 18 points and Part B is worth 12 points. Remember to print your name and student ID in the spaces provided in both Part A and Part B.

The points for each problem are indicated in the right margin.

Permitted resources: none. No books, no notes of any kind, no calculators, no electronic devices of any kind.

This is a midterm test. Cheating on an test is considered a serious offense by the University and can be met with disciplinary action, including suspension or expulsion. Candidates shall not bring into the test room any books, resources or papers except at the discretion of the examiner or as indicated on the examination paper. Candidates shall hold no communication of any kind with other candidates within the examination room.

PRINT your NAME and STUDENT ID: ____________________

PART A. Fully answer the following questions in the space provided.

Question A1. Find the value of a such that $y = -x + \tan x + a$ is a solution of the differential equation $\frac{dy}{dx} = (x+y+1)^2$. 3

$\frac{dy}{dx} = -1 + \sec^2 x = \tan^2 x \qquad (x+y+1)^2 = (\tan x + a + 1)^2$

$\tan^2 x = (\tan x + a + 1)^2$ for all x $\Rightarrow a = -1$, eg put $x = 0$.

Question A2. Find the solution of the initial value problem $\frac{dy}{dx} = \frac{1+x^2}{1+y}$, $y(1) = -2$. Leave your answer as an implicit equation for y. 3

$\int (1+y)\,dy = \int (1+x^2)\,dx$

$y + \frac{1}{2}y^2 = x + \frac{1}{3}x^3 + C$

$y(1) = -2 \Rightarrow -2+2 = 1+\frac{1}{3}+C = \frac{4}{3}+C$

$C = \frac{-4}{3}$

$y + \frac{1}{2}y^2 = x + \frac{1}{3}x^3 - \frac{4}{3}$

Question A3 Find the general solution of the first order linear equation $\frac{dy}{dx} + \frac{2}{x}y = \frac{e^x}{x}$ 3

$I = \exp\int P(x)\,dx = \exp\int \frac{2}{x}\,dx = \exp(2\ln x) = x^2$

$\frac{d}{dx}\left(x^2 y\right) = \frac{e^x}{x}\cdot x^2 = xe^x \qquad x^2 y = \int xe^x\,dx + C = xe^x - e^x + C$

$y = \frac{1}{x^2}\left((x-1)e^x + C\right) = \frac{x-1}{x^2}e^x + \frac{C}{x^2}$. C could be different on $x>0$ and $x<0$

Question A4. Find the general solution for each of the three differential equations 3

$$\frac{d^2y}{dx^2} + 8\frac{dy}{dx} + 15y = 0, \qquad 4\frac{d^2y}{dx^2} - 12\frac{dy}{dx} + 9y = 0, \qquad \frac{d^2y}{dx^2} - 2\frac{dy}{dx} + 2y = 0$$

I: $r^2+8r+15 = (r+3)(r+5)$ so $y = Ae^{-3x} + Be^{-5x}$

II: $4r^2-12r+9 = (2r-3)^2$ so $y = (Ax+B)e^{\frac{3}{2}x}$

III: $r^2-2r+2 = 0 \Rightarrow r = \frac{2\pm\sqrt{4-8}}{2} = 1\pm i$ so $y = e^x(A\cos x + B\sin x)$

Question A5. Simplify, to the form $a + bi$ where a and b are real, the expression 3

$$z = 2 + i + \frac{(3-i)(2-i)}{1+2i}.$$

$\frac{(3-i)(2-i)}{1+2i} = \frac{5-5i}{1+2i} = \frac{(5-5i)(1-2i)}{5} = \frac{-5-15i}{5} = -1-3i$

$z = 2+i-1-3i = 1-2i$

Question A6. Find the solution of the initial value problem $\frac{d^2y}{dx^2} + 4\frac{dy}{dx} + 3y = e^{2x}$, $y(0) = \cancel{1}\ \frac{1}{15}$, $y'(0)=1$ 3

$r^2+4r+3 = (r+3)(r+1) = 0 \quad r=-3$ or $r=-1 \quad y_h = A_1e^{-3x} + A_2e^{-x}$

$y_p = Be^{2x} \quad y_p'' + 4y_p' + 3y_p = Be^{2x}(4+4\cdot 2+3) = Be^{2x}(15) = e^{2x}$

$B = \frac{1}{15} \quad y = A_1e^{-3x} + A_2e^{-x} + \frac{1}{15}e^{2x}$

$y(0)=1 \Rightarrow A_1 + A_2 + \frac{1}{15} = \frac{1}{15} \qquad A_1 + A_2 = 0 \qquad A_1 = \frac{13}{30},\ A_2 = \frac{13}{30}$

$y'(0)=0 \Rightarrow -3A_1 - A_2 + \frac{2}{15} = 1 \qquad 3A + A_2 = \frac{-13}{15}$

$y = \frac{-13}{30}e^{-3x} + \frac{13}{30}e^{-x} + \frac{1}{15}e^{2x}$

Math 223 Test #2 PART B. (02, G.Patrick)

PART B Fully answer the following questions in the space provided.

PRINT your NAME and STUDENT ID: ______________________

Question B1. The following two questions refer to the differential equation 4

$$a_0\frac{d^ny}{dx^n} + a_1\frac{d^{n-1}y}{dx^{n-1}} + \cdots + a_ny = G(x),$$

auxillary

where $a_0, \ldots, a_n$ are numbers. For the given the characteristic equation and the given function $G(x)$, write the form of the particular solution that you would use to solve the equations by the method of undetermined coefficients.

a. Characteristic equation $(r-1)^2(r-3-2i)^4(r-3+2i)^4 = 0$, and $G(x) = e^{3x}\sin 2x$

b. Characteristic equation $r^3(r-1)^2 = 0$, and $G(x) = 1 + xe^{-x}$

a) $e^{3x}\sin 2x \to 3+2i$ and this has multiplicity 4. So
$y_p = x^4e^{3x}(A\cos 2x + B\sin 2x)$

b) $1 \to 0+0i$, which has multiplicity 3 $\to Ax^3$

$e^{-x} \to -1+0i$ which has multiplicity 0 $\to e^{-x}(B_0+B_1x)$

$y_p = Ax^3 + e^{-x}(B_0+B_1x)$

Question B2. Find the general solution of the differential equation $\frac{dy}{dx} = \frac{x}{y} + \frac{y}{x} + 1$. 4
leave as implicit

It is homogeneous. So put $y = ux$, $\frac{dy}{dx} = u + x\frac{du}{dx}$

$u + x\frac{du}{dx} = \frac{1}{u} + u + 1$

$x\frac{du}{dx} = \frac{1}{u} + 1 = \frac{1+u}{u}$ $\quad \int\frac{u\,du}{1+u} = \int\frac{dx}{x}$ $\quad \frac{u}{1+u} = \frac{1+u-1}{1+u} = 1 - \frac{1}{1+u}$

$u - \ln|1+u| = \ln|x| + C$

$\frac{y}{x} - \ln\left|1+\frac{y}{x}\right| = \ln|x| + C \quad \frac{y}{x} - \ln|x+y| = C$

Question B3. Usually, initial value problems for ordinary differential equations have unique solutions, but in fact there are some exceptions to this statement. Find two distinct solutions of the initial value problem $\frac{dy}{dx} = \frac{4}{3}y^{1/4}$, $y(0) = 0$.

Its separable. $\int \frac{dy}{y^{1/4}} = \int y^{-1/4} dy = \frac{1}{1-\frac{1}{4}} y^{1-\frac{1}{4}} = \frac{4}{3} y^{3/4}$

so $\frac{4}{3} y^{3/4} = \frac{4}{3} x + C$ $y = 0$ when $x = 0 \Rightarrow C = 0$

$y = x^{\frac{4}{3}}$ is a solution

but $y = 0$ is another solution, since it satisfies

$\frac{dy}{dx} = \frac{4}{3} y^{1/4}$ by direct substitution